

AFFINE QUASI-HEREDITY OF AFFINE SCHUR ALGEBRAS

BANGMING DENG AND GUIYU YANG

ABSTRACT. In this paper we prove that the affine Schur algebra $\widehat{S}(n, r)$ is affine quasi-hereditary. This result is then applied to show that $\widehat{S}(n, r)$ has finite global dimension and its centralizer subquotient algebras are Laurent polynomial algebras. We also use the result to give a parameter set of simple $\widehat{S}(n, r)$ -modules and identify this parameter set with that given in [4].

1. INTRODUCTION

Affine (quantum) Schur algebras play a central role in linking the representations of affine quantum groups and affine Hecke algebras. These algebras can be defined in several equivalent ways and has been widely studied; see, for example, [11, 12, 16, 20, 4].

The notion of affine quasi-hereditary algebras, which are an affine analogue of quasi-hereditary algebras, was introduced in [15] as a description of a kind of algebras which have affine Hecke algebras of type A as a perfect model. In [13], affine quasi-hereditary (graded) algebras was systematically studied and an affine analogue of the Cline–Parshall–Scott Theorem was given. In particular, quiver Hecke algebras (or KLR algebras) of finite type and Kato’s geometric extension algebras are shown to be affine quasi-hereditary algebras in [13].

It was shown in [1] that affine quantum Schur algebra $\widehat{S}_v(n, r)$ are affine cellular in the sense of König–Xi [15] by proving that the affine cell ideals of $\widehat{S}_v(n, r)$ are generated by canonical bases introduced by Lusztig in [16] and can be viewed as generalized matrix algebras over the representation ring of certain direct product of general linear groups. Recently, Nakajima [19] proved that the cell ideals of modified quantum affine algebras are idempotent, and, in particular, the BLN algebras are affine cellular with idempotent cell ideals. As a kind of BLN algebra, $\widehat{S}_v(n, r)$ has idempotent cell ideals in case $n > r$.

The aim of this paper is to prove that the affine Schur algebra $\widehat{S}(n, r)$ (by evaluating the parameter $v = 1$) is affine quasi-hereditary in the sense of [15]. In case $n > r$, we identify the cell ideals given in [1] with the ideals generated by some idempotents in $\widehat{S}(n, r)$. This identification gives the affine quasi-heredity of $\widehat{S}(n, r)$. In case $n \leq r$, we use Schur functors to derive an affine-heredity chain of $\widehat{S}(n, r)$, which affords an affine quasi-hereditary structure for $\widehat{S}(n, r)$ in this case. As applications, we prove that $\widehat{S}(n, r)$ has finite global dimension and its centralizer subquotient algebras of $\widehat{S}(n, r)$ are Laurent polynomial algebras. Furthermore,

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we use the affine-heredity chain to give a parameter set of simple $\widehat{S}(n, r)$ -modules and identify this parameter set with that given in [4].

In a forthcoming paper, we will consider the affine quasi-heredity of affine quantum Schur algebras.

The paper is organized as follows. In Sections 2–4 we recall the definitions of affine cellular algebras and affine quasi-hereditary algebras and review two definitions of the affine Schur algebra $\widehat{S}(n, r)$ and a construction of its affine cell chain. The affine quasi-heredity of $\widehat{S}(n, r)$ in case $n > r$ is proved in Section 5. We then use the Schur functor to prove the affine quasi-heredity of $\widehat{S}(n, r)$ in case $n \leq r$ in Section 6. As an application, we prove in Section 7 that the centralizer subquotient algebras of $\widehat{S}(n, r)$ are Laurent polynomial algebras. In the final section, we give a parameter set of simple $\widehat{S}(n, r)$ -modules and identify it with the parameter set given in [4].

2. AFFINE CELLULAR ALGEBRAS AND AFFINE QUASI-HEREDITARY ALGEBRAS

In this section we recall the definitions of affine cellular algebras and affine quasi-hereditary algebras introduced in [15] and [13], respectively. For a module over an algebra, we always mean a left module unless otherwise stated.

Suppose that k is a Noetherian domain. A commutative k -algebra B is called *affine* if it is a quotient of a polynomial ring $k[x_1, \dots, x_s]$ in finitely many variables x_1, \dots, x_s by an ideal I , i.e. $B = k[x_1, \dots, x_s]/I$. A k -linear anti-automorphism τ of a k -algebra A with $\tau^2 = \text{id}_A$ will be called a k -involution on A .

Definition 2.1 ([15]). *Let A be a unitary k -algebra with a k -linear involution τ on A . A two-sided ideal J in A is called an affine cell ideal if the following conditions are satisfied:*

- (1) $\tau(J) = J$.
- (2) *There is a free k -module V of finite rank and an affine algebra B with a k -involution σ such that $\Delta = V \otimes_k B$ is an A - B -bimodule, where the right B -module structure is induced by that of the regular right B -module B_B .*
- (3) *There is an A - A -bimodule isomorphism $\alpha : J \rightarrow \Delta \otimes_B \Delta'$, where $\Delta' = B \otimes_k V$ is a B - A -bimodule with the left B -structure induced by ${}_B B$ and the right A -module structure is given as $(b \otimes v)a = p(\tau(a)(v \otimes b))$, where p is the switch map:*

$$p : \Delta \otimes_B \Delta' \longrightarrow \Delta' \otimes_B \Delta,$$

$$x \otimes y \longrightarrow y \otimes x, \text{ for } x \in \Delta \text{ and } y \in \Delta'.$$

- (4) *There is the following commutative diagram:*

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \\ \downarrow \tau & & \downarrow v_1 \otimes b_1 \otimes_B b_2 \otimes v_2 \rightarrow v_2 \otimes \sigma(b_2) \otimes_B \sigma(b_1) \otimes v_1 \\ J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \end{array}$$

The algebra A is called affine cellular if and only if there is a k -module decomposition $A = J'_1 \oplus J'_2 \oplus \dots \oplus J'_n$ with $\tau(J'_j) = J'_j$ for each j , and $J_i = \bigoplus_{1 \leq l \leq i} J'_l$ gives a chain of two-sided ideals of A : $0 = J_0 \subset J_1 \subset J_2 \subset \dots \subset J_n = A$, and for each $1 \leq i \leq n$, $J'_i = J_i/J_{i-1}$ is an affine cell ideal of A/J_{i-1} .

We call this chain an affine cell chain for the affine cellular algebra A . The bimodule Δ will be called a cell lattice for the affine cell ideal J .

The following two lemmas are taken from [15, Lem. 2.4] and [15, Th. 4.3].

Lemma 2.2. *Suppose that K is another Noetherian doamin and $\psi : k \rightarrow K$ is a homomorphism of rings with identity. If A is an affine cellular k -algebra, then $K \otimes_k A$ is an affine cellular K -algebra with the canonical affine cell chain induced from that of k -algebra A .*

Lemma 2.3. *Let $J = V \otimes_k B \otimes_k V$ be an idempotent affine cell ideal in a k -algebra A with the cell lattice $\Delta = V \otimes_k B$ as in Definition 2.1.*

(1) *If there is a nonzero idempotent e in J , then ${}_A J$ is a projective A -module, $J = AeA$, and there is an equality $\text{add}({}_A Ae) = \text{add}({}_A \Delta)$.*

(2) *If $\text{rad}(B) = 0$, then $\text{End}({}_A \Delta) \cong B$.*

Now we give the definition of an affine quasi-hereditary algebra following [15, 13].

Definition 2.4. *Let A be a left Noetherian k -algebra. An ideal J in A is called an affine-heredity ideal if the following conditions are satisfied:*

- (1) *There is an idempotent $e \in J$ such that $J = AeA$;*
- (2) *As a left A -module, $J \cong \underbrace{P \oplus \cdots \oplus P}_m$, for some projective A -module P and some $m \in \mathbb{N}$, and $B := \text{End}_A(P)$ is an affine algebra, which means that B is a finitely generated commutative algebra.*
- (3) *As a right B -module, P is finitely generated and flat.*

The algebra A is called affine quasi-hereditary if there is a finite chain of ideals

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_t = A$$

such that J_i/J_{i-1} is an affine-heredity ideal of A/J_{i-1} for each $1 \leq i \leq t$. Such a chain of ideals is called an affine-heredity chain.

Remark 2.5. The definition for an affine quasi-hereditary given here follows from the one in [15]. A graded version is given in [13].

3. AFFINE SCHUR ALGEBRAS

This section is devoted to introducing the geometric definition of affine (quantum) Schur algebras given by [11, 16]. An algebraic definition of affine Schur algebras given in [21] will also be reviewed. Finally, we give a correspondence of the basis elements between these two definitions and recall some multiplication formulas in [4, 16, 21].

Let \mathbb{F} be a field and let $\mathbb{F}[x, x^{-1}]$ be the Laurent polynomial ring in indeterminate x . Fix an $\mathbb{F}[x, x^{-1}]$ -free module V of rank $r \geq 1$. A lattice in V is a free $\mathbb{F}[x]$ -submodule L of V such that $V = L \otimes_{\mathbb{F}[x]} \mathbb{F}[x, x^{-1}]$.

Let $\mathfrak{F}_\Delta = \mathfrak{F}_{\Delta, n}$ be the set of all cyclic flags $L = (L_i)_{i \in \mathbb{Z}}$ of lattices, where each L_i is a lattice in V such that $L_{i-1} \subseteq L_i$ and $L_{i-n} = xL_i$ for all $i \in \mathbb{Z}$. The group G of automorphisms of the the $\mathbb{F}[x, x^{-1}]$ -module V acts on \mathfrak{F}_Δ by $g \cdot L = (g(L_i))_{i \in \mathbb{Z}}$ for $g \in G$ and $L \in \mathfrak{F}_\Delta$. Thus, the map

$$(3.1) \quad \phi : \mathfrak{F}_\Delta \longrightarrow \Lambda_\Delta(n, r), \quad L \longmapsto \underline{\dim} L = (\dim_{\mathbb{F}} L_i / L_{i-1})_{i \in \mathbb{Z}}$$

induces a bijection between the set of G -orbits in \mathfrak{F}_Δ and $\Lambda_\Delta(n, r)$, where

$$\Lambda_\Delta(n, r) := \{(\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{N}, \sum_{i=1}^n \lambda_i = r \text{ and } \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}\}.$$

We denote by \mathfrak{F}_λ the fiber of λ under this map ϕ , i.e. $\mathfrak{F}_\lambda = \phi^{-1}(\lambda)$.

Let

$$\Lambda(n, r) := \{(\lambda_1, \dots, \lambda_n) \mid \lambda_i \in \mathbb{N}, \sum_{1 \leq i \leq n} \lambda_i = r\}.$$

We usually identify $\Lambda(n, r)$ with $\Lambda_\Delta(n, r)$ via the following bijection:

$$b : \Lambda_\Delta(n, r) \longrightarrow \Lambda(n, r), \quad \lambda \longmapsto (\lambda_1, \dots, \lambda_n).$$

The group G also acts diagonally on $\mathfrak{F}_\Delta \times \mathfrak{F}_\Delta$ by $g(L, L') = (gL, gL')$, where $g \in G$ and $L, L' \in \mathfrak{F}_\Delta$. By [16, 1.5], there is a bijection between the set of G -orbits in $\mathfrak{F}_\Delta \times \mathfrak{F}_\Delta$ and the set $\Theta_\Delta(n, r)$ by sending (L, L') to $A = (a_{i,j})_{i,j \in \mathbb{Z}}$, where

$$(3.2) \quad a_{i,j} = \dim_{\mathbb{F}} \frac{L_i \cap L'_j}{L_{i-1} \cap L'_j + L_i \cap L'_{j-1}} \quad \text{for } i, j \in \mathbb{Z},$$

$$(3.3) \quad \Theta_\Delta(n, r) := \{A = (a_{i,j})_{i,j \in \mathbb{Z}} \in M_{\Delta,n}(\mathbb{N}) \mid \sum_{\substack{1 \leq i \leq n \\ j \in \mathbb{Z}}} a_{i,j} = \sum_{\substack{1 \leq j \leq n \\ i \in \mathbb{Z}}} a_{i,j} = r\}$$

and $M_{\Delta,n}(\mathbb{N})$ is the set of all $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ with $a_{i,j} \in \mathbb{N}$ such that

- (a) $a_{i,j} = a_{i+n,j+n}$ for $i, j \in \mathbb{Z}$;
- (b) for every $i \in \mathbb{Z}$, the set $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$ is finite.

Let \mathcal{O}_A denote the orbit in $\mathfrak{F}_\Delta \times \mathfrak{F}_\Delta$ corresponding to A . If $(L, L') \in \mathcal{O}_A$, then $\text{row}(A) = \underline{\dim} L$ and $\text{col}(A) = \underline{\dim} L'$, where

$$\text{row}(A) = \left(\sum_{j \in \mathbb{Z}} a_{i,j} \right)_{i \in \mathbb{Z}}, \quad \text{col}(A) = \left(\sum_{i \in \mathbb{Z}} a_{i,j} \right)_{j \in \mathbb{Z}}.$$

Assume now that $\mathbb{F} = \mathbb{F}_q$ is a finite field of q elements and write $\mathfrak{F}_\Delta(q)$ for \mathfrak{F}_Δ . Suppose that $A, A', A'' \in \Theta_\Delta(n, r)$. For any fixed $(L, L'') \in \mathcal{O}_{A''}$, let

$$c_{A,A',A'';q} = |\{L' \in \mathfrak{F}_\Delta(q) \mid (L, L') \in \mathcal{O}_A, (L', L'') \in \mathcal{O}_{A'}\}|.$$

Clearly, $c_{A,A',A'';q}$ is independent of the choice of (L, L'') , and a necessary condition for $c_{A,A',A'';q} \neq 0$ is that

$$(3.4) \quad \text{col}(A) = \text{row}(A'), \quad \text{row}(A) = \text{row}(A'') \text{ and } \text{col}(A') = \text{col}(A'').$$

By [16], there is a polynomial $p_{A,A',A''} \in \mathbb{Z}[v, v^{-1}]$ in v^2 such that for each finite field \mathbb{F}_q with q elements, $c_{A,A',A'';q} = p_{A,A',A''}|_{v^2=q}$.

Definition 3.1 ([16],[11]). *The affine quantum Schur algebra $\widehat{S}_v(n, r)$ is the free $\mathbb{Z}[v, v^{-1}]$ -module with basis $\{e_A \mid A \in \Theta_\Delta(n, r)\}$, and multiplication defined by*

$$e_A \cdot e_{A'} = \begin{cases} \sum_{A'' \in \Theta_\Delta(n, r)} p_{A,A',A''} e_{A''}, & \text{if } \text{col}(A) = \text{row}(A'), \\ 0, & \text{otherwise.} \end{cases}$$

As in the finite case, for each $\lambda \in \Lambda_\Delta(n, r)$, define $\text{diag}(\lambda) = (\delta_{i,j} \lambda_i)_{i,j \in \mathbb{Z}} \in \Theta_\Delta(n, r)$, and $\mathbf{l}_\lambda = e_{\text{diag}(\lambda)}$. It is easy to see that for each $A \in \Theta_\Delta(n, r)$,

$$(3.5) \quad \mathbf{l}_\lambda e_A = \begin{cases} e_A, & \text{if } \lambda = \text{row}(A); \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad e_A \mathbf{l}_\lambda = \begin{cases} e_A, & \text{if } \lambda = \text{col}(A); \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\sum_{\lambda \in \Lambda_\Delta(n, r)} \mathbf{l}_\lambda$ is the identity of $\widehat{S}_v(n, r)$.

For each ring R which is a $\mathbb{Z}[v, v^{-1}]$ -module, we set

$$\widehat{S}(n, r)_R := \widehat{S}_v(n, r) \otimes_{\mathbb{Z}[v, v^{-1}]} R.$$

In particular, for the ring of integers \mathbb{Z} and any field \mathbb{F} , we have

$$\widehat{S}(n, r) = \widehat{S}(n, r)_{\mathbb{Z}} \quad \text{and} \quad \widehat{S}(n, r)_{\mathbb{F}},$$

where \mathbb{Z} and \mathbb{F} are viewed as a $\mathbb{Z}[v, v^{-1}]$ -module by specializing v to 1.

Now we introduce the algebraic definition of affine Schur algebras given in [21]. Let \mathfrak{S}_r denote the symmetric group on r letters and let $\mathfrak{S}_\Delta = \mathfrak{S}_r \ltimes \mathbb{Z}^r$ denote the extended affine Weyl group of type \widehat{A}_{r-1} . Define

$$I(n, r) = \{\underline{i} = (i_1, \dots, i_r) \mid 1 \leq i_t \leq n, \text{ for } 1 \leq t \leq r\},$$

$$I(\mathbb{Z}, r) = \{\underline{i} = (i_1, \dots, i_r) \mid i_t \in \mathbb{Z}, \text{ for } 1 \leq t \leq r\}.$$

The group \mathfrak{S}_r acts on $I(n, r)$ by place permutation while \mathfrak{S}_Δ acts on $I(\mathbb{Z}, r)$ on the right with \mathfrak{S}_r acting by place permutation and \mathbb{Z}^r acting by shifting, i.e.,

$$\underline{i}(\sigma, \varepsilon) = \underline{i} + n\varepsilon,$$

for $\underline{i} \in I(\mathbb{Z}, r)$, $\sigma \in \mathfrak{S}_r$ and $\varepsilon \in \mathbb{Z}^r$. Then \mathfrak{S}_Δ acts diagonally on $I(\mathbb{Z}, r) \times I(\mathbb{Z}, r)$. For $(\underline{i}, \underline{j}) \in I(\mathbb{Z}, r) \times I(\mathbb{Z}, r)$ and $(\underline{k}, \underline{l}) \in I(\mathbb{Z}, r) \times I(\mathbb{Z}, r)$, we identify $\xi_{\underline{i}, \underline{j}}$ and $\xi_{\underline{k}, \underline{l}}$ if and only if $(\underline{i}, \underline{j}) \sim_{\mathfrak{S}_\Delta} (\underline{k}, \underline{l})$, i.e., $(\underline{i}, \underline{j})$ and $(\underline{k}, \underline{l})$ are in the same orbit.

Definition 3.2 ([21]). *The affine Schur algebra $\widehat{S}'(n, r)$ is defined to be the \mathbb{Z} -algebra with basis $\{\xi_{\underline{i}, \underline{j}} \mid \underline{i}, \underline{j} \in I(\mathbb{Z}, r) / \sim_{\mathfrak{S}_\Delta}\}$ and multiplication given by the following rule:*

$$\xi_{\underline{i}, \underline{j}} \xi_{\underline{k}, \underline{l}} = \sum_{(\underline{p}, \underline{q}) \in I(\mathbb{Z}, r) \times I(\mathbb{Z}, r) / \mathfrak{S}_\Delta} C(\underline{i}, \underline{j}, \underline{k}, \underline{l}, \underline{p}, \underline{q}) \xi_{\underline{p}, \underline{q}},$$

where $C(\underline{i}, \underline{j}, \underline{k}, \underline{l}, \underline{p}, \underline{q}) = |\{\underline{s} \in I(\mathbb{Z}, r) \mid (\underline{i}, \underline{j}) \sim_{\mathfrak{S}_\Delta} (\underline{p}, \underline{s}), (\underline{s}, \underline{q}) \sim_{\mathfrak{S}_\Delta} (\underline{k}, \underline{l})\}|$.

Remark 3.3. There is a \mathbb{Z} -algebra isomorphism

$$\varphi: \widehat{S}'(n, r) \longrightarrow \widehat{S}(n, r) = \widehat{S}(n, r)_{\mathbb{Z}}, \quad \xi_{\underline{i}, \underline{j}} \longmapsto e_A = e_A \otimes 1,$$

where $\underline{i}, \underline{j} \in I(\mathbb{Z}, r)$ and $A = (a_{x,y})_{x,y \in \mathbb{Z}} \in \Theta_\Delta(n, r)$ is defined by setting

$$a_{x,y} = \sharp\{s \mid i_s = x, j_s = y, 1 \leq s \leq r\}.$$

In the following we will simply identify $\widehat{S}'(n, r)$ with $\widehat{S}(n, r)$. We now give some useful multiplication formulas from [21, 22].

Proposition 3.4 ([21]). *Let $\underline{i}, \underline{j}, \underline{k}, \underline{l} \in I(\mathbb{Z}, r)$. We have the following equalities in $S_\Delta(n, r)$.*

- (1) $\xi_{\underline{i}, \underline{j}} \xi_{\underline{k}, \underline{l}} = 0$ unless $\underline{j} \sim_{\mathfrak{S}_\Delta} \underline{k}$.
- (2) $\xi_{\underline{i}, \underline{i}} \xi_{\underline{i}, \underline{j}} = \xi_{\underline{i}, \underline{j}} = \xi_{\underline{i}, \underline{j}} \xi_{\underline{i}, \underline{i}}$.

(3) $\sum_{\underline{i} \in I(n,r)/\mathfrak{S}_r} \xi_{\underline{i},\underline{i}}$ is a decomposition of unity into orthogonal idempotents.

Proposition 3.5 ([21], [22]). (1)

$$\begin{aligned} \xi_{\underline{i},\underline{j}} \xi_{\underline{j},\underline{l}} &= \sum_{\delta \in \mathfrak{S}_{\Delta,\underline{j},\underline{l}} \setminus \mathfrak{S}_{\Delta,\underline{j}} / \mathfrak{S}_{\Delta,\underline{i},\underline{j}}} \left[\mathfrak{S}_{\Delta,\underline{i},\underline{l}\delta} : \mathfrak{S}_{\Delta,\underline{i},\underline{j},\underline{l}\delta} \right] \xi_{\underline{i},\underline{l}\delta} \\ &= \sum_{\delta \in \mathfrak{S}_{\Delta,\underline{i},\underline{j}} \setminus \mathfrak{S}_{\Delta,\underline{j}} / \mathfrak{S}_{\Delta,\underline{j},\underline{l}}} \left[\mathfrak{S}_{\Delta,\underline{i}\delta,\underline{l}} : \mathfrak{S}_{\Delta,\underline{i}\delta,\underline{j},\underline{l}} \right] \xi_{\underline{i}\delta,\underline{l}}, \end{aligned}$$

where $\underline{i}, \underline{j}, \underline{l}$ are in $I(\mathbb{Z}, r)$, $\mathfrak{S}_{\Delta,\underline{i}}$ is the stabilizer subgroup of \underline{i} in \mathfrak{S}_{Δ} and $\mathfrak{S}_{\Delta,\underline{i},\underline{j}}$ is the stabilizer of \underline{i} and \underline{j} in \mathfrak{S}_{Δ} , i.e. $\mathfrak{S}_{\Delta,\underline{i},\underline{j}} = \mathfrak{S}_{\Delta,\underline{i}} \cap \mathfrak{S}_{\Delta,\underline{j}}$, etc, $\mathfrak{S}_{\Delta,\underline{j},\underline{l}} \setminus \mathfrak{S}_{\Delta,\underline{j}} / \mathfrak{S}_{\Delta,\underline{i},\underline{j}}$ denotes a representative set of double cosets.

(2)

$$\begin{aligned} \xi_{\underline{i},\underline{j}+n\varepsilon} \xi_{\underline{j},\underline{l}+n\varepsilon'} &= \sum_{\delta \in \mathfrak{S}_{\underline{j},\underline{l},\varepsilon'} \setminus \mathfrak{S}_{\underline{j}} / \mathfrak{S}_{\underline{i},\underline{j},\varepsilon}} \left[\mathfrak{S}_{\underline{i},\underline{l}\delta,\varepsilon'+\varepsilon} : \mathfrak{S}_{\underline{i},\underline{j},\underline{l}\delta,\varepsilon'+\varepsilon} \right] \xi_{\underline{i},\underline{l}\delta+n(\varepsilon'+\varepsilon)} \\ &= \sum_{\delta \in \mathfrak{S}_{\underline{i},\underline{j},\varepsilon} \setminus \mathfrak{S}_{\underline{j}} / \mathfrak{S}_{\underline{j},\underline{l},\varepsilon'}} \left[\mathfrak{S}_{\underline{i}\delta,\underline{l},\varepsilon'+\varepsilon\delta} : \mathfrak{S}_{\underline{i}\delta,\underline{j},\underline{l},\varepsilon'+\varepsilon\delta} \right] \xi_{\underline{i}\delta,\underline{l}+n(\varepsilon'+\varepsilon\delta)} \end{aligned}$$

where $\underline{i}, \underline{j}, \underline{l}$ are in $I(n, r)$, $\varepsilon, \varepsilon' \in \mathbb{Z}^r$, $\mathfrak{S}_{\underline{i}}$ is the stabilizer subgroup of \underline{i} in \mathfrak{S}_r and $\mathfrak{S}_{\underline{i},\underline{j}}$ is the stabilizer of \underline{i} and \underline{j} in \mathfrak{S}_r , i.e. $\mathfrak{S}_{\underline{i},\underline{j}} = \mathfrak{S}_{\underline{i}} \cap \mathfrak{S}_{\underline{j}}$, etc, $\mathfrak{S}_{\underline{j},\underline{l},\varepsilon'} \setminus \mathfrak{S}_{\underline{j}} / \mathfrak{S}_{\underline{i},\underline{j},\varepsilon}$ denotes a representative set of double cosets.

4. CANONICAL BASIS AND TWO-SIDED CELLS OF $\widehat{S}_v(n, r)$

In this section we introduce the canonical basis of $\widehat{S}_v(n, r)$ given in [16] and its two-sided cells given in [17]. We finally give an affine cell chain of $\widehat{S}_v(n, r)$ constructed in [1].

For $A = (a_{i,j}) \in \Theta_{\Delta}(n, r)$, let

$$[A] = v^{-d_A} e_A, \quad \text{where } d_A = \sum_{\substack{1 \leq i \leq n \\ i \geq k, j < l}} a_{i,j} a_{k,l}.$$

Suppose that $A, A_1 \in \Theta_{\Delta}(n, r)$ satisfy

$$\text{row}(A) = \text{row}(A_1) = \lambda \quad \text{and} \quad \text{col}(A) = \text{col}(A_1) = \lambda'.$$

Fix a cyclic flag $L \in \mathfrak{F}_{\lambda}$, define $A_1 \leq A$ if $X_{A_1}^L \subset \bar{X}_A^L$, where

$$X_A^L = \{L' \in \mathfrak{F}_{\lambda'} \mid (L, L') \in \mathcal{O}_A\}.$$

Lusztig has defined in [16, Sect. 4] the canonical basis $\{\{A\} \mid A \in \Theta_{\Delta}(n, r)\}$ of $\widehat{S}_v(n, r)$ by

$$(4.1) \quad \{A\} = \sum_{A_1: A_1 \leq A} \Pi_{A_1, A} [A_1],$$

where $\Pi_{A,A} = 1$ and $\Pi_{A_1, A} \in v^{-1}\mathbb{Z}[v^{-1}]$ if $A_1 < A$.

Recall that for $\lambda \in \Lambda^+(n, r)$, $\mathfrak{l}_{\lambda} = e_{\text{diag}(\lambda)}$. We use $\{\mathfrak{l}_{\lambda}\}$ to denote the canonical basis $\{\text{diag}(\lambda)\}$.

Lemma 4.1. For $\lambda \in \Lambda^+(n, r)$, we have $\{\mathfrak{l}_{\lambda}\} = \mathfrak{l}_{\lambda}$ in $\widehat{S}_v(n, r)$.

Proof. Fix a cyclic flag $L \in \mathfrak{F}_\lambda$. Since $\mathfrak{l}_\lambda = e_{\text{diag}(\lambda)}$, it follows that

$$X_{\text{diag}(\lambda)}^L = \{L' \in \mathfrak{F}_\lambda \mid (L, L') \in \mathcal{O}_{\text{diag}(\lambda)}\} = \{L\}.$$

Applying (4.1) implies that the only term in $\{\mathfrak{l}_\lambda\}$ with nonzero coefficient is \mathfrak{l}_λ , that is, $\{\mathfrak{l}_\lambda\} = \mathfrak{l}_\lambda$. □

Definition 4.2 ([17]). For $A, A' \in \Theta_\Delta(n, r)$, let

$$\{A\}\{A'\} = \sum_{A''} \nu_{A, A'}^{A''} \{A''\},$$

where $\nu_{A, A'}^{A''} \in \mathbb{Z}[v, v^{-1}]$.

We say that $A \preceq_{LR} A'$ if there is a sequence $A' = A_1, A_2, \dots, A_m = A \in \Theta_\Delta(n, r)$ and a sequence $B_1, \dots, B_{m-1} \in \Theta_\Delta(n, r)$ such that $\nu_{B_s, A_s}^{A_{s+1}} \neq 0$ or $\nu_{A_s, B_s}^{A_{s+1}} \neq 0$ for $1 \leq s \leq m-1$. We write $A \sim_{LR} A'$ if $A \preceq_{LR} A'$ and $A' \preceq_{LR} A$. The equivalence classes of \sim_{LR} are called two-sided cells of $\widehat{S}_v(n, r)$.

Recall that $\Lambda(n, r) := \{\lambda = (\lambda_1, \dots, \lambda_n) \mid \lambda_i \in \mathbb{N}, \sum_{1 \leq i \leq n} \lambda_i = r\}$ and

$$\Lambda^+(n, r) := \{\lambda \in \Lambda(n, r) \mid \lambda_1 \geq \dots \geq \lambda_n\}.$$

Proposition 4.3 ([17]). Let $A \in \Theta_\Delta(n, r)$. An antidiagonal path in A is an infinite strip of entries $(a_{i_k, j_k} : k \in \mathbb{Z})$ such that (i_k, j_k) is obtained from (i_{k-1}, j_{k-1}) by subtracting 1 from the first entry or adding 1 to the second, with the latter being the case for all but finitely many k . Thus, visually if you draw the matrix with rows increasing from top to bottom and columns from left to right (as we will do), then path starts and ends with infinite vertical strips, and takes finitely many right or vertical turns.

Let d_j be the maximal size of the sum of entries in the union of j antidiagonal paths. Then, we define a map

$$(4.2) \quad \begin{aligned} \rho : \Theta_\Delta(n, r) &\longrightarrow \Lambda^+(n, r), \\ A &\longmapsto (d_1, d_2 - d_1, \dots, d_n - d_{n-1}). \end{aligned}$$

Define

$$\mathfrak{c}_\lambda = \{\{A\} \mid A \in \rho^{-1}(\lambda)\}.$$

Then $\{\mathfrak{c}_\lambda \mid \lambda \in \Lambda^+(n, r)\}$ are the two-sided cells of $\widehat{S}_v(n, r)$.

Lemma 4.4. For $\lambda \in \Lambda^+(n, r)$, we have $\{\mathfrak{l}_\lambda\} \in \mathfrak{c}_\lambda$.

Proof. This can be directly checked by using Proposition 4.3. □

Let “ \geq ” be a partial ordering on $\Lambda^+(n, r)$ defined by $\lambda \geq \mu$ whenever $\sum_{j=1}^i \lambda_j \geq \sum_{j=1}^i \mu_j$, for $1 \leq i \leq n$. It is obvious that $(r, 0, \dots, 0)$ is the maximal element with respect to this ordering. We then fix a total ordering $\lambda^{(1)} > \lambda^{(2)} > \dots > \lambda^{(t)}$ on $\Lambda^+(n, r)$ which is a refinement of the above ordering, where t equals the number of elements in $\Lambda^+(n, r)$.

For each $1 \leq i \leq t$, define

$$\mathfrak{c}_i = \{\{A\} \mid A \in \rho^{-1}(\lambda^{(i)})\}.$$

Then $\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_t$ are the two sided cells in $\widehat{S}_v(n, r)$ such that $\mathfrak{c}_i \preceq_{LR} \mathfrak{c}_j$ implies $i \leq j$.

For $1 \leq i \leq t$, let C'_i be the $\mathbb{Z}[v, v^{-1}]$ -submodule of $\widehat{S}_v(n, r)$ generated by all $\{e_A\}$ with $A \in \mathfrak{c}_i$. Let $C_i = \oplus_{l=1}^i C'_l$. Then by [1], C_i is an ideal of $\widehat{S}_v(n, r)$ for each $1 \leq i \leq t$ and the chain of ideals

$$(4.3) \quad 0 \subseteq C_1 \subseteq \dots \subseteq C_i \subseteq \dots \subseteq C_t = \widehat{S}_v(n, r)$$

forms an affine cell chain of $\widehat{S}_v(n, r)$. This implies particularly that $\widehat{S}_v(n, r)$ is an affine cellular algebra over $\mathbb{Z}[v, v^{-1}]$.

Now applying Lemma 2.2 to the evaluation map $\mathbb{Z}[v, v^{-1}] \rightarrow \mathbb{Z}$ taking $v^\pm \mapsto 1$, we conclude that $\widehat{S}(n, r)$ is an affine cellular \mathbb{Z} -algebra which admits an affine cell chain induced from (4.3). Thus, in the following sections we simply view the C_i as ideals of $\widehat{S}(n, r)$ and the chain (4.3) as an affine cell chain of $\widehat{S}(n, r)$.

5. AFFINE QUASI-HEREDITY OF $\widehat{S}(n, r)$ IN CASE $n > r$

In this section we prove that $\widehat{S}(n, r)$ is affine quasi-hereditary under the assumption $n > r$. Keep all the notations in the previous sections.

Since $\widehat{S}(n, r)$ is a kind of BLN algebra when $n > r$, it follows from [19] (see also [2]) that affine cell ideals of $\widehat{S}(n, r)$ are idempotent ideals.

Proposition 5.1 ([19], [2]). *Suppose that $n > r$. Then for each $1 \leq i \leq t$, $C'_i = C_i/C_{i-1}$ is an idempotent ideal of $\widehat{S}(n, r)/C_{i-1}$.*

Fix a total ordering $\lambda^{(1)} > \lambda^{(2)} > \dots > \lambda^{(t)}$ on $\Lambda^+(n, r)$ as in Section 4, where t equals the number of elements in $\Lambda^+(n, r)$.

For each $1 \leq i \leq t$, set

$$J_i = \widehat{S}(n, r) \left(\sum_{\mu \in \Lambda^+(n, r), \mu \geq \lambda^{(i)}} \mathfrak{l}_\mu \right) \widehat{S}(n, r).$$

Lemma 5.2. *Assume $n > r$. Then $J_i = C_i$ for each $1 \leq i \leq t$. Moreover, $\bar{J}_i = J_i/J_{i-1}$ is projective as an \bar{S} -module, where $\bar{S} = \widehat{S}(n, r)/J_{i-1}$.*

Proof. First we remark that $\lambda^{(1)} = (r, 0, \dots, 0) \in \Lambda^+(n, r)$. By Lemmas 4.1 and 4.4,

$$\mathfrak{l}_{\lambda^{(1)}} = \{\mathfrak{l}_{\lambda^{(1)}}\} \in C'_1.$$

Since C_1 is an idempotent ideal of $\widehat{S}(n, r)$ by Proposition 5.1, it follows from Lemma 2.3 that

$$C_1 = \widehat{S}(n, r) \mathfrak{l}_{\lambda^{(1)}} \widehat{S}(n, r) = J_1,$$

and J_1 is projective as a left $\widehat{S}(n, r)$ -module.

For each $1 \leq i \leq t$, the above analysis is valid for $\bar{S} = \widehat{S}(n, r)/J_{i-1}$. Then we get that

$$(5.1) \quad C'_i = C_i/C_{i-1} = \bar{S} \mathfrak{l}_{\lambda^{(i)}} \bar{S},$$

and C_i/C_{i-1} is projective as a left \bar{S} -module.

Now we prove that $C_i = J_i$ for $2 \leq i \leq t$ by induction. By Lemmas 4.1 and 4.4,

$$\sum_{1 \leq l \leq i} \mathfrak{l}_{\lambda^{(l)}} = \sum_{1 \leq l \leq i} \{\mathfrak{l}_{\lambda^{(l)}}\} \in C_i.$$

Then there is a canonical inclusion from J_i to C_i for $1 \leq i \leq t$. This together with the induction hypothesis induces an inclusion of \mathbb{Z} -modules

$$\theta : J_i/J_{i-1} \hookrightarrow C_i/J_{i-1} = C_i/C_{i-1}$$

taking $\bar{I}_{\lambda^{(i)}} \mapsto \bar{I}_{\lambda^{(i)}}$. It is easy to check that

$$J_i/J_{i-1} \cong \bar{S}\bar{I}_{\lambda^{(i)}}\bar{S}.$$

On the other hand, $C_i/C_{i-1} = \bar{S}\bar{I}_{\lambda^{(i)}}\bar{S}$ by (5.1). Then θ is an isomorphism. This proves that $J_i = C_i$. \square

Now we introduce the representation ring of the general linear group. It has been shown in [1] that the affine cell ideals of $\widehat{S}(n, r)$ can be considered as generalized matrix algebras over these representation rings.

Let $R(\mathrm{GL}_m(\mathbb{C}))$ denote the representation ring of $\mathrm{GL}_m(\mathbb{C})$, where $m \in \mathbb{N}$. It is shown in [10, Exercise 23.36] that

$$R(\mathrm{GL}_m(\mathbb{C})) \cong \mathbb{Z}[x_1, x_2, \dots, x_m, x_m^{-1}].$$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n, r)$, denote by $R(\prod_{l=1}^n \mathrm{GL}_{\lambda(l)}(\mathbb{C}))$ the representation ring of $\prod_{l=1}^n \mathrm{GL}_{\lambda(l)}(\mathbb{C})$, where $\lambda(l) = \lambda_l - \lambda_{l+1}$ with $\lambda_{n+1} = 0$. Define

$$(5.2) \quad B_\lambda = R\left(\prod_{l=1}^n \mathrm{GL}_{\lambda(l)}(\mathbb{C})\right).$$

Then

$$B_\lambda = \mathbb{Z}[x_1, x_2, \dots, x_{\lambda_1}, x_{m(1)}^{-1}, x_{m(2)}^{-1}, \dots, x_{m(n)}^{-1}],$$

where $m(i) = \lambda_1 - \lambda_{i+1}$ and $\lambda_{n+1} = 0$. For example, if $\lambda = (4, 2, 1)$, then

$$B_\lambda = \mathbb{Z}[x_1, x_2, x_2^{-1}, x_3, x_3^{-1}, x_4, x_4^{-1}].$$

Theorem 5.3. *In case $n > r$, $\widehat{S}(n, r)$ is an affine hereditary \mathbb{Z} -algebra.*

Proof. By [23, Th. 5], $\widehat{S}(n, r)$ is Noetherian. Recall that $\lambda^{(1)} > \lambda^{(2)} > \dots > \lambda^{(t)}$ is a total ordering on the elements of $\Lambda^+(n, r)$ and

$$J_i = \widehat{S}(n, r) \left(\sum_{\mu \in \Lambda^+(n, r), \mu \geq \lambda^{(i)}} I_\mu \right) \widehat{S}(n, r),$$

for $1 \leq i \leq t$. Then we obtain the following chain of ideals of $\widehat{S}(n, r)$:

$$0 = J_0 \subseteq J_1 \subseteq \dots \subseteq J_t = \widehat{S}(n, r).$$

It suffices to prove that J_i/J_{i-1} is an affine-heredity ideal of $\bar{S} = \widehat{S}(n, r)/J_{i-1}$ for $1 \leq i \leq t$. By Lemma 5.2, $J_i/J_{i-1} = \bar{S}\bar{I}_{\lambda^{(i)}}\bar{S}$ is an idempotent ideal of \bar{S} and $\bar{S}\bar{I}_{\lambda^{(i)}}\bar{S}$ is projective over \bar{S} . Thus, condition (1) in Definition 2.4 is fulfilled.

By Definition 2.1 and Lemma 5.2, for each $1 \leq i \leq t$, there are \bar{S} - $B_{\lambda^{(i)}}$ -bimodule Δ and $B_{\lambda^{(i)}}$ - \bar{S} -bimodule Δ' with the following \bar{S} -bimodule isomorphism

$$\alpha : J_i/J_{i-1} \longrightarrow \Delta \otimes_{B_{\lambda^{(i)}}} \Delta',$$

where $B_{\lambda^{(i)}}$ is defined as in (5.2).

For notational simplicity, we write B_i for $B_{\lambda^{(i)}}$. We remark that Δ and Δ' are free of the same finite rank over B_i when viewed as right and left modules, respectively.

Since $J_i/J_{i-1} = \bar{S}\bar{\mathfrak{l}}_{\lambda(i)}\bar{S}$ is generated by the idempotent $\bar{\mathfrak{l}}_{\lambda(i)}$, it follows from Lemma 2.3 that

$$\text{add}(\bar{S}\bar{\mathfrak{l}}_{\lambda(i)}) = \text{add}(\bar{S}\Delta).$$

This implies that Δ is projective as a left \bar{S} -module. Since Δ' is a free B_i -module of finite rank, we get the following decomposition of $\bar{S}\bar{\mathfrak{l}}_{\lambda(i)}\bar{S}$ as a direct sum of projective \bar{S} -modules:

$$\bar{S}\bar{\mathfrak{l}}_{\lambda(i)}\bar{S} = J_i/J_{i-1} \cong \underbrace{\Delta \oplus \dots \oplus \Delta}_m,$$

where m equals the rank of Δ' over B_i .

To prove that $\bar{S}\bar{\mathfrak{l}}_{\lambda(i)}\bar{S}$ satisfies condition (2) of Definition 2.4, we only need to show that $\text{End}_{\bar{S}}(\Delta)$ is an affine algebra. Indeed, by the definition of B_i , it is easy to see that $\text{rad}(B_i) = 0$. Then, by Lemma 2.3, $\text{End}_{\bar{S}}(\Delta) \cong B_i$ is an affine algebra.

Since Δ is free of finite rank over B_i , condition (3) in Definition 2.4 is satisfied. This finishes the proof. \square

6. AFFINE QUASI-HEREDITY OF $\widehat{S}(n, r)$ IN CASE $n \leq r$

In this section we prove that $\widehat{S}(n, r)$ is affine quasi-hereditary when $n \leq r$. We first give some properties of $\widehat{S}(n, r)$ which will be needed in the proof. Note that these properties are valid for $\widehat{S}(n, r)$ with n, r arbitrary.

Recall that there is an action of the symmetric group \mathfrak{S}_n on $\Lambda(n, r)$ given by

$$(6.1) \quad w \cdot (\lambda_1, \dots, \lambda_n) = (\lambda_{w(1)}, \dots, \lambda_{w(n)}),$$

where $w \in \mathfrak{S}_n$.

Lemma 6.1. *Let $\lambda, \mu \in \Lambda(n, r)$. If there is some $w \in \mathfrak{S}_n$ such that $w \cdot \lambda = \mu$, then*

$$\widehat{S}(n, r)\mathfrak{l}_\lambda\widehat{S}(n, r) = \widehat{S}(n, r)\mathfrak{l}_\mu\widehat{S}(n, r).$$

Proof. By [22, Lem. 2], $\widehat{S}(n, r)\mathfrak{l}_\lambda \cong \widehat{S}(n, r)\mathfrak{l}_\mu$. The isomorphism is induced by two elements $X, Y \in \widehat{S}(n, r)$ such that

$$X \in \mathfrak{l}_\lambda\widehat{S}(n, r)\mathfrak{l}_\mu, \quad Y \in \mathfrak{l}_\mu\widehat{S}(n, r)\mathfrak{l}_\lambda, \quad XY = \mathfrak{l}_\lambda, \quad \text{and} \quad YX = \mathfrak{l}_\mu.$$

Since $X\mathfrak{l}_\mu = X$ and $Y\mathfrak{l}_\lambda = Y$, we get that

$$X\mathfrak{l}_\mu Y = XY = \mathfrak{l}_\lambda \quad \text{and} \quad Y\mathfrak{l}_\lambda X = YX = \mathfrak{l}_\mu.$$

This implies that \mathfrak{l}_λ is contained in the ideal generated by \mathfrak{l}_μ , and vice versa. Therefore,

$$\widehat{S}(n, r)\mathfrak{l}_\lambda\widehat{S}(n, r) = \widehat{S}(n, r)\mathfrak{l}_\mu\widehat{S}(n, r).$$

\square

Lemma 6.2. *Write $\widehat{S} = \widehat{S}(n, r)$. Then*

$$\widehat{S} = \widehat{S}\left(\sum_{\lambda \in \Lambda(n, r)} \mathfrak{l}_\lambda\right)\widehat{S} = \widehat{S}\left(\sum_{\lambda \in \Lambda^+(n, r)} \mathfrak{l}_\lambda\right)\widehat{S}.$$

Proof. Since

$$\sum_{\lambda \in \Lambda(n, r)} \mathfrak{l}_\lambda = \text{id},$$

we get that

$$\widehat{S} = \widehat{S} \left(\sum_{\lambda \in \Lambda(n, r)} \mathfrak{l}_\lambda \right) \widehat{S}.$$

By (6.1), each orbit of $\Lambda(n, r)$ under the action of \mathfrak{S}_n has exactly one element in $\Lambda^+(n, r)$. It follows from Lemma 6.1 that

$$\widehat{S} = \widehat{S} \left(\sum_{\lambda \in \Lambda(n, r)} \mathfrak{l}_\lambda \right) \widehat{S} = \widehat{S} \left(\sum_{\lambda \in \Lambda^+(n, r)} \mathfrak{l}_\lambda \right) \widehat{S},$$

as desired. \square

Now we are ready to prove the main result in this section. Choose $N > r$. Then by Theorem 5.3, $\widehat{S}(N, r)$ is affine quasi-hereditary with an affine-heredity chain

$$(6.2) \quad 0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_i \subseteq \cdots \subseteq J_t = \widehat{S}(N, r),$$

where t is the number of elements in $\Lambda^+(N, r)$ and

$$J_i = \widehat{S}(N, r) \left(\sum_{\mu \in \Lambda^+(N, r), \mu \geq \lambda^{(i)}} \mathfrak{l}_\mu \right) \widehat{S}(N, r).$$

Since $n \leq r < N$, there is an inclusion map from $\Lambda(n, r)$ to $\Lambda(N, r)$

$$\begin{aligned} \Lambda(n, r) &\longrightarrow \Lambda(N, r) \\ (\lambda_1, \dots, \lambda_n) &\longmapsto (\lambda_1, \dots, \lambda_n, 0, \dots, 0). \end{aligned}$$

Then we can view $\Lambda(n, r)$ as a subset of $\Lambda(N, r)$. Furthermore, let

$$e = \sum_{\lambda \in \Lambda(n, r)} \mathfrak{l}_\lambda.$$

Then $e^2 = e \in \widehat{S}(N, r)$ and it is easily checked that

$$(6.3) \quad e \widehat{S}(N, r) e \cong \widehat{S}(n, r).$$

Lemma 6.3. *Let $\{J_i \mid 1 \leq i \leq t\}$ be the ideals of $\widehat{S}(N, r)$ in the affine quasi-heredity chain as given in (6.2). Then there is some $1 \leq l \leq t$ such that $J_l = \widehat{S}(N, r) e \widehat{S}(N, r)$.*

Proof. Set

$$x = \sum_{\lambda \in \Lambda^+(n, r)} \mathfrak{l}_\lambda.$$

Then there is some $1 \leq l \leq m$ such that $J_l = \widehat{S}(N, r) x \widehat{S}(N, r)$. By Lemma 6.1,

$$\widehat{S}(N, r) e \widehat{S}(N, r) = \widehat{S}(N, r) x \widehat{S}(N, r) = J_l.$$

\square

Theorem 6.4. *The algebra $e \widehat{S}(N, r) e$ is affine quasi-hereditary. In other words, $\widehat{S}(n, r) \cong e \widehat{S}(N, r) e$ is affine quasi-hereditary when $n \leq r$.*

Proof. Take the affine-heredity chain of $\widehat{S}(N, r)$ as in (6.2)

$$0 = J_0 \subseteq J_1 \subseteq \cdots J_l \subseteq \cdots \subseteq J_t = \widehat{S}(N, r)$$

such that $J_l = \widehat{S}(N, r)e\widehat{S}(N, r)$, where $e = \sum_{\lambda \in \Lambda(n, r)} \mathfrak{l}_\lambda$.

We want to show that

$$0 = eJ_0e \subseteq eJ_1e \subseteq \cdots eJ_l e \subseteq \cdots \subseteq eJ_t e = e\widehat{S}(N, r)e$$

is an affine-heredity chain of $e\widehat{S}(N, r)e$, i.e., $\widehat{S}(n, r) = e\widehat{S}(N, r)e$ is affine quasi-hereditary.

First we prove that $eJ_l e = e\widehat{S}(N, r)e$. To simplify the notation, we use \widehat{S} to denote $\widehat{S}(N, r)$ in what follows. Since $J_l = \widehat{S}e\widehat{S}$, we deduce that

$$eJ_l e = e\widehat{S}e\widehat{S}e = e(\widehat{S}e)^2 = e\widehat{S}e.$$

Now we only need to prove that eJ_1e is an affine-hereditary ideal of $e\widehat{S}e$, the other requirements follow easily by induction.

We first check that $eJ_1e = (eJ_1e)^2$. It is obvious that $eJ_1e \supseteq (eJ_1e)^2$. Note that $J_1 = \widehat{S}\mathfrak{l}_{\lambda^{(1)}}\widehat{S}$, where $\lambda^{(1)} = (r, 0, \dots, 0) \in \Lambda^+(N, r)$. Since

$$\begin{aligned} (eJ_1e)^2 &= eJ_1eJ_1e \\ &= e\widehat{S}\mathfrak{l}_{\lambda^{(1)}}\widehat{S}e\widehat{S}\mathfrak{l}_{\lambda^{(1)}}\widehat{S}e && \text{by } J_1 = \widehat{S}\mathfrak{l}_{\lambda^{(1)}}\widehat{S} \\ &\supseteq e\widehat{S}\mathfrak{l}_{\lambda^{(1)}}\widehat{S}e && \text{by } \mathfrak{l}_{\lambda^{(1)}} \in \widehat{S}e\widehat{S} \\ &= eJ_1e, \end{aligned}$$

we get that $eJ_1e = (eJ_1e)^2$. This proves that eJ_1e satisfies condition (1) of Definition 2.4.

Now we want to show that eJ_1e can be decomposed into a direct sum of some projective $e\widehat{S}e$ -module P and that the endomorphism algebra of P is an affine algebra.

Since J_1 is an affine cell ideal of \widehat{S} , there are \widehat{S} - B -bimodule Δ and B - \widehat{S} -bimodule Δ' with the following \widehat{S} - \widehat{S} -bimodule isomorphism

$$J_1 \longrightarrow \Delta \otimes_B \Delta',$$

where $B = B_{\lambda^{(1)}}$, Δ and Δ' are free over B of finite rank. By [22, Prop. 6],

$$B = B_{\lambda^{(1)}} \cong \mathbb{Z}[x_1, \dots, x_r, x_r^{-1}].$$

Since J_1 is idempotent and contains a nonzero idempotent $\mathfrak{l}_{\lambda^{(1)}}$, we get by Lemma 2.3 that

$$\text{add}(\widehat{S}\mathfrak{l}_{\lambda^{(1)}}) = \text{add}(\Delta).$$

Then Δ is projective as a left \widehat{S} -module and J_1 can be decomposed into a direct sum of projective \widehat{S} -modules

$$J_1 \cong \underbrace{\Delta \oplus \cdots \oplus \Delta}_s,$$

where s is the rank of Δ' over B .

Further, by [24, Lem. 3.3],

$$(6.4) \quad eJ_1e \longrightarrow (e\Delta) \otimes_B (\Delta'e)$$

is an $e\widehat{S}e$ -bimodule isomorphism which makes eJ_1e an affine cell ideal of $e\widehat{S}e$. Since eJ_1e is idempotent and contains a nonzero idempotent $\mathbf{l}_{\lambda(1)}$, it follows from Lemma 2.3 that

$$(6.5) \quad \text{add}_{e\widehat{S}e} \left((e\widehat{S}e)\mathbf{l}_{\lambda(1)} \right) = \text{add}_{e\widehat{S}e} \left(e\Delta \right).$$

Hence, $e\Delta$ is a left projective $e\widehat{S}e$ -module. Since $\Delta'e$ is free of finite rank over B , we obtain an $e\widehat{S}e$ -module decomposition

$$eJ_1e = \underbrace{e\Delta \oplus \dots \oplus e\Delta}_a,$$

where a is the rank of $\Delta'e$ over B .

Since $\text{rad}(B) = 0$, we get that $\text{End}_{e\widehat{S}e}(e\Delta) \cong B$ by Lemma 2.3. Thus, $\text{End}_{e\widehat{S}e}(e\Delta)$ is an affine algebra. Consequently, eJ_1e satisfies condition (2) of Definition 2.4.

Since Δ is free of finite rank over B as a right module, it is easy to check that $e\Delta$ is projective and finitely generated over B . Thus, condition (3) of Definition 2.4 is satisfied. Hence, eJ_1e is an affine-hereditary ideal of $e\widehat{S}e$. The proof is completed. \square

Theorem 6.4 together with Theorem 5.3 gives a positive answer to a conjecture in [13] and [1] which states that $\widehat{S}(n, r)$ is affine quasi-hereditary. Furthermore, the affine-heredity chain of $\widehat{S}(n, r)$ satisfies the conditions in [15, Th. 4.4]. i.e., for each $1 \leq i \leq t$, $\text{rad}(B_{\lambda(i)}) = 0$, $\bar{S}\bar{\mathbf{l}}_{\lambda(i)}\bar{S}$ is idempotent and contains a nonzero idempotent in \bar{S} . We remark that in this case, the global dimension of $\widehat{S}(n, r)$ is finite if and only if the global dimension of $B_{\lambda(i)}$ is finite for every $1 \leq i \leq t$. Since $B_{\lambda(i)}$ is a localization of $\mathbb{Z}[x_1, \dots, x_{\lambda_1}]$, it has finite global dimension. Therefore, we obtain the following corollary.

Corollary 6.5. *For all positive integers n and r , the global dimension of $\widehat{S}(n, r)$ is finite.*

7. APPLICATION I: A DESCRIPTION OF SUBQUOTIENT ALGEBRAS OF $\widehat{S}(n, r)_{\mathbb{Q}}$

In this section we always work with the \mathbb{Q} -algebra $\widehat{S}(n, r)_{\mathbb{Q}}$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n, r)$, set

$$(7.1) \quad B_{\lambda} = (B_{\lambda})_{\mathbb{Q}} = R \left(\prod_{l=1}^n \text{GL}_{\lambda(l)}(\mathbb{C}) \right) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\lambda(l) = \lambda_l - \lambda_{l+1}$ and $\lambda_{n+1} = 0$. i.e.

$$B_{\lambda} = \mathbb{Q}[x_1, x_2, \dots, x_{\lambda_1}, x_{m(1)}^{-1}, x_{m(2)}^{-1}, \dots, x_{m(n)}^{-1}],$$

where $m(i) = \lambda_1 - \lambda_{i+1}$ and $\lambda_{n+1} = 0$.

The main aim of this section is to prove that for each $\lambda \in \Lambda^+(n, r)$,

$$\bar{\mathbf{l}}_{\lambda} \bar{S} \bar{\mathbf{l}}_{\lambda} \cong B_{\lambda},$$

where $\bar{S} = \widehat{S}(n, r)_{\mathbb{Q}}/J$ with $J = \widehat{S}(n, r)_{\mathbb{Q}}(\sum_{\mu \in \Lambda^+(n, r), \mu > \lambda} \mathbf{l}_{\mu})\widehat{S}(n, r)_{\mathbb{Q}}$.

First we have the following lemma.

Lemma 7.1. *For each $\lambda \in \Lambda^+(n, r)$, $\bar{\mathbf{l}}_{\lambda} \bar{S} \bar{\mathbf{l}}_{\lambda}$ is Morita equivalent to B_{λ} .*

Proof. It is clear that $\bar{l}_\lambda \bar{S} \bar{l}_\lambda \cong \text{End}_{\bar{S}}(\bar{S} \bar{l}_\lambda)$. Since $\bar{S} \bar{l}_\lambda \bar{S}$ is an affine cell ideal of \bar{S} , there are \bar{S} - B_λ -bimodule Δ and B_λ - \bar{S} -bimodule Δ' with the following \bar{S} - \bar{S} -bimodule isomorphism

$$\bar{S} \bar{l}_\lambda \bar{S} \longrightarrow \Delta \otimes_{B_\lambda} \Delta'.$$

Then $\text{add}_{\bar{S}}(\bar{S} \bar{l}_\lambda) = \text{add}_{\bar{S}}(\Delta)$ by Lemma 2.3. This implies that $\bar{l}_\lambda \bar{S} \bar{l}_\lambda$ and $\text{End}_{\bar{S}}(\Delta)$ are Morita equivalent. Since $\text{rad}(B_\lambda) = 0$, we get that $\text{End}_{\bar{S}}(\Delta) \cong B_\lambda$ by Lemma 2.3. This proves the lemma. \square

By the commutativity of B_λ and Lemma 7.1, in order to get the isomorphism $\bar{l}_\lambda \bar{S} \bar{l}_\lambda \cong B_\lambda$, it suffices to prove that $\bar{S} \bar{l}_\lambda \bar{S}$ is commutative. To prove this fact, we need the following basic results on classical Schur algebra $S(n, r)_\mathbb{Q}$ (over \mathbb{Q}).

We first recall a refined triangular decomposition and a presentation of $S(n, r)_\mathbb{Q}$ given in [8] and [7], respectively. For $1 \leq i, j \leq n$, let $E_{i,j} = (a_{k,l})$ be the $n \times n$ elementary matrix in \mathbb{N} defined by

$$a_{k,l} = \begin{cases} 1, & \text{if } k = i, l = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$e_i = \sum_{\lambda \in \Lambda(n, r-1)} e_{E_{i,i+1} + \text{diag}(\lambda)}, \quad f_i = \sum_{\lambda \in \Lambda(n, r-1)} e_{E_{i+1,i} + \text{diag}(\lambda)}, \quad l_\lambda = e_{\text{diag}(\lambda)}$$

for $1 \leq i < n$ and $\lambda \in \Lambda(n, r)$. Suppose that I is the set of finite sequence $B = (i_1, \dots, i_m)$ (including the empty sequence \emptyset), with each i_l for $1 \leq l \leq m$ contained in the set $\{1, 2, \dots, n-1\}$. For each sequence $B = (i_1, \dots, i_m)$, define

$$e_B = e_{i_1} e_{i_2} \cdots e_{i_m}; \quad f_B = f_{i_m} \cdots f_{i_2} f_{i_1}$$

and set $e_\emptyset = f_\emptyset = 1$ by convention.

Lemma 7.2 ([8, Prop 2.7]). *Let $S(n, r)_\mathbb{Q}$ be the classical Schur algebra over \mathbb{Q} . Then $S(n, r)_\mathbb{Q}$ is spanned by products of the form $f_B e_\lambda e_D$, where B, D are in I and $\lambda \in \Lambda(n, r)$.*

Lemma 7.3 ([7, Thm 1.4]). *The \mathbb{Q} -algebra $S(n, r)$ is generated by*

$$e_i, f_i, l_\lambda \quad (1 \leq i \leq n-1, \lambda \in \Lambda(n, r))$$

subject to the following relations

- (1) $l_\lambda l_\mu = \delta_{\lambda, \mu} l_\lambda, \quad \sum_{\lambda \in \Lambda(n, r)} l_\lambda = 1,$
- (2) $e_i l_\lambda = \begin{cases} l_{\lambda + \alpha_i - \alpha_{i+1}} e_i, & \text{if } \lambda_{i+1} \geq 1, \\ 0, & \lambda_{i+1} = 0. \end{cases}$
- (3) $l_\lambda e_i = \begin{cases} e_i l_{\lambda - \alpha_i + \alpha_{i+1}}, & \text{if } \lambda_i \geq 1, \\ 0, & \lambda_i = 0. \end{cases}$
- (4) $f_i l_\lambda = \begin{cases} l_{\lambda - \alpha_i + \alpha_{i+1}} f_i, & \text{if } \lambda_i \geq 1, \\ 0, & \lambda_i = 0. \end{cases}$
- (5) $l_\lambda f_i = \begin{cases} f_i l_{\lambda + \alpha_i - \alpha_{i+1}}, & \text{if } \lambda_{i+1} \geq 1, \\ 0, & \lambda_{i+1} = 0. \end{cases}$
- (6) $e_i f_j - f_j e_i = \delta_{i,j} \sum_{\lambda \in \Lambda(n, r)} (\lambda_i - \lambda_{i+1}) l_\lambda,$

$$\begin{aligned}
 (7) \quad & e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 = 0, \quad (|i - j| = 1) \\
 & e_i e_j = e_j e_i \quad (\text{otherwise}) \\
 (8) \quad & f_i^2 f_j - 2f_i f_j f_i + f_j f_i^2 = 0, \quad (|i - j| = 1) \\
 & f_i f_j = f_j f_i \quad (\text{otherwise})
 \end{aligned}$$

where $\alpha_i = (0, \dots, \underset{(i)}{1}, \dots, 0) \in \mathbb{Z}^n$.

Theorem 7.4. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n, r)$. Then $\bar{\mathfrak{l}}_\lambda \bar{S} \bar{\mathfrak{l}}_\lambda \cong B_\lambda$.*

Proof. We identify \mathfrak{l}_λ with $\xi_{\underline{i}, \underline{i}}$, where $\underline{i} = (\underbrace{1, \dots, 1}_{\lambda_1}, \dots, \underbrace{n, \dots, n}_{\lambda_n}) \in I(n, r)$. Then $\mathfrak{l}_\lambda \widehat{S}(n, r)_\mathbb{Q} \mathfrak{l}_\lambda$ is spanned by the set

$$\mathfrak{Y} = \{e_A = \xi_{\underline{i}, \underline{i}\sigma + n\varepsilon} \mid \sigma \in \mathfrak{S}_r, \varepsilon \in \mathbb{Z}^r\}.$$

Let

$$\mathfrak{X} = \{e_A = \xi_{\underline{i}, \underline{i}\sigma} \mid \sigma \in \mathfrak{S}_r\}$$

be a subset of \mathfrak{Y} . This implies that

$$\mathfrak{X} = \{e_A \mid A = (a_{i,j})_{i,j \in \mathbb{Z}} \in \Theta_\Delta(n, r), \text{col}(A) = \text{row}(A) = \lambda, \sum_{1 \leq s, t \leq n} a_{s,t} = r\}.$$

In this sense, we can view \mathfrak{X} as a subset of $\mathfrak{l}_\lambda S(n, r) \mathfrak{l}_\lambda$. By Lemma 7.2, for each $e_A \in \mathfrak{X}$, we can write

$$e_A = \sum_s a_s (f_{B_s} \mathfrak{l}_{\lambda^{(s)}} e_{D_s}),$$

where $a_s \in \mathbb{Q}$, $B_s, D_s \in I$ and $\lambda^{(s)} \in \Lambda(n, r)$. Consequently,

$$e_A = \mathfrak{l}_\lambda e_A \mathfrak{l}_\lambda = \sum_s \mathfrak{l}_\lambda f_{B_s} \mathfrak{l}_{\lambda^{(s)}} e_{D_s} \mathfrak{l}_\lambda.$$

If $D_s \neq \emptyset$ and $e_{D_s} \mathfrak{l}_\lambda \neq 0$ for some s , then

$$e_{D_s} \mathfrak{l}_\lambda = \mathfrak{l}_{\lambda'} e_{D_s},$$

where $\lambda' > \lambda$ by Lemma 7.3. This means that $\mathfrak{l}_\lambda f_{B_s} e_{\lambda^{(s)}} e_{D_s} \mathfrak{l}_\lambda \in J$. Similarly, if $B_s \neq \emptyset$ and $\mathfrak{l}_\lambda f_{B_s} \neq 0$ for some s , then $\mathfrak{l}_\lambda f_{B_s} \mathfrak{l}_{\lambda^{(s)}} e_{D_s} \mathfrak{l}_\lambda \in J$. Therefore, for each $A \in \mathfrak{X}$,

$$\bar{e}_A = \overline{\mathfrak{l}_\lambda e_A \mathfrak{l}_\lambda} = \sum_s \overline{\mathfrak{l}_\lambda \mathfrak{l}_{\lambda^{(s)}} \mathfrak{l}_\lambda} = a \bar{\mathfrak{l}}_\lambda,$$

where a equals the number of s such that $\lambda^{(s)} = \lambda$. Hence, $\bar{\mathfrak{l}}_\lambda \bar{S} \bar{\mathfrak{l}}_\lambda$ is \mathbb{Q} -spanned by

$$(7.2) \quad \mathfrak{M} = \{\bar{e}_A = \bar{\xi}_{\underline{i}, \underline{i} + n\varepsilon} \mid \varepsilon \in \mathbb{Z}^r\}.$$

Let Λ be the subalgebra of $\mathfrak{l}_\lambda \widehat{S}(n, r)_\mathbb{Q} \mathfrak{l}_\lambda$ which is \mathbb{Q} -spanned by $\{\xi_{\underline{i}, \underline{i} + n\varepsilon} \mid \varepsilon \in \mathbb{Z}^r\}$. Applying [22, Prop. 8] gives

$$\Lambda \cong \mathbb{C}[x_1, x_2, \dots, x_r, x_1^{-1}, x_2^{-1}, \dots, x_\alpha^{-1}],$$

where α is the number of nonzero entries in $\{\lambda_1, \dots, \lambda_n\}$. Then by (7.2), we get an epimorphism from Λ to $\bar{\mathfrak{l}}_\lambda \bar{S} \bar{\mathfrak{l}}_\lambda$. This proves that $\bar{\mathfrak{l}}_\lambda \bar{S} \bar{\mathfrak{l}}_\lambda$ is a commutative algebra. Applying Lemma 7.1 gives $\bar{\mathfrak{l}}_\lambda \bar{S} \bar{\mathfrak{l}}_\lambda \cong B_\lambda$. This proves the theorem. \square

8. APPLICATION II: AN IDENTIFICATION OF PARAMETER SETS OF SIMPLE MODULES

In this section we give a parameter set of simple $\widehat{S}(n, r)_{\mathbb{C}}$ -modules and further identify this parameter set with the set given in [4] and [9].

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n, r)$, recall that

$$B_{\lambda} = \mathbb{C}[x_1, x_2, \dots, x_{\lambda_1}, x_{m(1)}^{-1}, x_{m(2)}^{-1}, \dots, x_{m(n)}^{-1}],$$

where $m(i) = \lambda_1 - \lambda_{i+1}$ and $\lambda_{n+1} = 0$. Define

$$\Omega_{\lambda} = \{\underline{a} = (a_1, a_2, \dots, a_{\lambda_1}) \in \mathbb{C}^{\lambda_1} \mid a_{m(i)} \neq 0, \text{ for } 1 \leq i \leq n\},$$

where $m(i) = \lambda_1 - \lambda_{i+1}$ and $\lambda_{n+1} = 0$.

For $\underline{a} = (a_1, a_2, \dots, a_{\lambda_1}) \in \Omega_{\lambda}$, define

$$E_{\underline{a}} = B_{\lambda} / (x_1 - a_1, \dots, x_{\lambda_1} - a_{\lambda_1}).$$

Then we have the following lemma.

Lemma 8.1. *For $\lambda \in \Lambda^+(n, r)$, the set*

$$\{E_{\underline{a}} \mid \underline{a} \in \Omega_{\lambda}\}$$

forms a complete set of non-isomorphic simple B_{λ} -modules.

Lemma 8.2. *Let $\Omega_{r,n} = \cup_{\lambda \in \Lambda^+(n,r)} \Omega_{\lambda}$. Then $\Omega_{r,n}$ is a parameter set of non-isomorphic simple $\widehat{S}(n, r)_{\mathbb{C}}$ -modules.*

Proof. Let

$$J_0 = 0 \subseteq J_1 \subseteq \dots \subseteq J_t = \widehat{S}(n, r)_{\mathbb{C}}$$

be the cell chain given in Theorem 5.3 and Theorem 6.4 such that $J_i/J_{i-1} \cong \bar{S}\bar{\Gamma}_{\lambda(i)}\widehat{S}(n, r)_{\mathbb{C}}$, where $\bar{S} = \widehat{S}(n, r)_{\mathbb{C}}/J_{i-1}$.

By [15, Cor. 3.2], simple $\widehat{S}(n, r)_{\mathbb{C}}$ -modules are parameterized by simple J_i/J_{i-1} -modules for all $1 \leq i \leq t$. Here for an ideal J_i/J_{i-1} of \bar{S} , a J_i/J_{i-1} -module M is called simple if $(J_i/J_{i-1})M \neq 0$ and there are no submodules different from 0 and M . Since $J_i/J_{i-1} \cong \bar{S}\bar{\Gamma}_{\lambda(i)}\bar{S}$, we get that the correspondence $L \mapsto \bar{\Gamma}_{\lambda(i)}L$ induces a bijection between the set of isomorphism classes of simple J_i/J_{i-1} -modules and that of isomorphism classes of simple $\bar{\Gamma}_{\lambda(i)}\bar{S}\bar{\Gamma}_{\lambda(i)}$ -modules.

By Lemma 7.4, $\bar{\Gamma}_{\lambda(i)}\bar{S}\bar{\Gamma}_{\lambda(i)} \cong B_{\lambda(i)}$. The lemma then follows from Lemma 8.1. \square

Now we recall the definition of segments in [4] (by specializing v to 1). A segment s is by definition a sequence

$$s = (a, a, \dots, a) \in (\mathbb{C}^*)^k,$$

where k is called the length of the segment, denoted by $|s|$. If $\mathbf{s} = \{s_1, \dots, s_p\}$ is an unordered collection of segments, define $\varrho(\mathbf{s})$ be the partition associated with the sequence $(|s_1|, \dots, |s_p|)$. That is, $\varrho(\mathbf{s}) = (|s_{i_1}|, \dots, |s_{i_p}|)$ with $|s_{i_1}| \geq \dots \geq |s_{i_p}|$, where $|s_{i_1}|, \dots, |s_{i_p}|$ is a permutation of $|s_1|, \dots, |s_p|$. We also call $|\mathbf{s}| = \sum_{1 \leq i \leq p} |s_i|$ the length of \mathbf{s} .

Let

$$C_{r,n} = \{\mathbf{s} = \{s_1, \dots, s_p\} \mid p \geq 1, \sum_{1 \leq i \leq p} |s_i| = r, |s_i| \leq n, \forall i, \}.$$

Then $C_{r,n} = \cup_{\lambda \in \Lambda^+(n,r)} C_{\lambda}$, where $C_{\lambda} = \{\mathbf{s} \in C_{r,n} \mid \varrho(\mathbf{s}) = \lambda\}$. Note that if $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n, r)$ and there is some $1 \leq i \leq n$ such that $\lambda_i > n$, then $C_{\lambda} = \emptyset$.

Lemma 8.3 ([4], [9]). $C_{r,n}$ is a parameter set of finite dimensional non-isomorphic simple $\widehat{S}(n, r)_{\mathbb{C}}$ -modules.

Proof. In [9], finite dimensional simple $\widehat{S}(n, r)_{\mathbb{C}}$ -modules are parameterized by the set of dominant polynomials. It is proved in [4, Thm. 4.4.2] and [3, Thm. 6.6] that the set of dominant polynomials and $C_{r,n}$ coincide. \square

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+(n, r)$, we define its dual partition λ' as follows

$$\lambda' = (\lambda'_1, \dots, \lambda'_n),$$

where $\lambda'_i = |\{j | \lambda_j \geq i\}|$, for $1 \leq i \leq n$. Note that for $\lambda \in \Lambda^+(n, r)$, its dual partition $\lambda' \in \Lambda^+(n, r)$ if and only if $\lambda_i \leq n$ for all $1 \leq i \leq n$.

Theorem 8.4. For arbitrary $r, n \in \mathbb{N}$, there is a bijection between $C_{r,n}$ and $\Omega_{r,n}$.

Proof. Take $\lambda = (\lambda_1, \dots, \lambda_i, 0, \dots, 0) \in \Lambda^+(n, r)$ with $\lambda_i \neq 0$ and $\lambda_j \leq n$ for $1 \leq j \leq i$. Then

$$\mathbf{s} = (\underbrace{a_1, \dots, a_1}_{\lambda_1}, \dots, \underbrace{a_i, \dots, a_i}_{\lambda_i}) \in C_{\lambda},$$

where $a_t \neq 0$ for $1 \leq t \leq i$.

Define

$$\begin{aligned} \phi : C_{r,n} &\longrightarrow \Omega_{r,n}, \\ \mathbf{s} = (\underbrace{a_1, \dots, a_1}_{\lambda_1}, \dots, \underbrace{a_i, \dots, a_i}_{\lambda_i}) &\longmapsto (b_1, \dots, b_i), \end{aligned}$$

where $b_{i_j} = e_{i_j}$ is the j -th elementary symmetric function in a_{i_1}, \dots, a_{i_s} if $\lambda_{i_1} = \dots = \lambda_{i_s}$ for $1 \leq i_1, \dots, i_s \leq i$.

It is easy to check that $\phi(\mathbf{s}) \in \Omega_{\lambda'}$ and ϕ is a bijection between $C_{r,n}$ and $\Omega_{r,n}$. The proof is completed. \square

Example 8.5. Let us consider some examples. Let $\lambda = (3, 2, 1)$. Then

$$C_{\lambda} = \{\underline{a} = (a_1, a_1, a_1, a_2, a_2, a_3) \mid a_i \in \mathbb{C}, \text{ for } 1 \leq i \leq 3, a_1 a_2 a_3 \neq 0\}.$$

Since $3 \neq 2 \neq 1$, by Theorem 8.4,

$$\phi(\underline{a}) = (a_1, a_2, a_3) \in \Omega_{\lambda'},$$

where $\lambda' = (3, 2, 1)$.

Let $\mu = (3, 3, 1)$. Then

$$C_{\mu} = \{\underline{a}' = (a_1, a_1, a_1, a_2, a_2, a_2, a_3) \mid a_i \in \mathbb{C}, \text{ for } 1 \leq i \leq 3, a_1 a_2 a_3 \neq 0\}.$$

Since $3 = 3 \neq 1$, by Theorem 8.4,

$$\phi(\underline{a}') = (a_1 + a_2, a_1 a_2, a_3) \in \Omega_{\mu'},$$

where $\mu' = (3, 2, 2)$. Let $\nu = (4, 2, 1)$. Since $\nu_1 = 4 > 3$, this implies that $C_{\nu} = \emptyset$. Its dual partition $\nu' = (3, 2, 1, 1)$ does not belong to $\Lambda^+(3, 7)$.

REFERENCES

- [1] W. Cui, *Affine cellularity of affine q -Schur algebras*, Arxiv: 14056705.
- [2] W. Cui, *Affine cellularity of BLN algebras*, Arxiv: 14056441.
- [3] B. Deng, J. Du, *Identification of simple representations for affine q -Schur algebras*, J. Algebra **373** (2013), 249–275.
- [4] B. Deng, J. Du, and Q. Fu, *A double Hall algebra approach to affine quantum Schur-Weyl theory*, London Math. Soc. Lecture Note Series 401, Cambridge University Press, Cambridge, 2012.
- [5] B. Deng, J. Du, B. Parshall and J. Wang, *Finite dimensional algebras and quantum groups*, Mathematical Surveys and Monographs Volume 150, Amer. Math. Soc., Providence 2008.
- [6] V. Dlab, C.M. Ringel, *Quasi-hereditary algebras*, Illinois J. Math. **33** (1989), 280–291.
- [7] S. Doty and A. Giaquinto, *Presenting Schur algebras*, Int. Math. Res. Not. **36**(2002), 1907–1944.
- [8] S. Doty and A. Giaquinto, *Cellular and quasihereditary structures of generalized quantized Schur algebras*, arxiv: 1012.5983v2.
- [9] Q. Fu, *Affine quantum Schur algebras at roots of unity*, arxiv: 1205.2997.
- [10] W. Fulton, J. Harris, *Representation theory*, GTM series **129** (1991).
- [11] V. Ginzburg and E. Vasserot, *Langlands reciprocity for affine quantum groups of type A_n* , Internat. Math. Res. Notices 1993, 67–85.
- [12] R. M. Green, *The affine q -Schur algebra*, J. Algebra. **215** (1999), 379–411.
- [13] A. S. Kleshchev, *Affine highest weight categories and affine quasihereditary algebras*, Arxiv: 14053328.
- [14] A. S. Kleshchev, J. W. Loubert, and V. Miemietz, *Affine cellularity of Khovanov-Lauda-Rouquier algebras in type A* , J. London Math. Soc. **88**(2013), no. 2, 338–358.
- [15] S. Koenig, C. C. Xi, *Affine cellular algebras*, Adv. Math., **229**(2012), 139–182.
- [16] G. Lusztig, *Aperiodicity in quantum affine $gl(n)$* , Asian J. Math. **3** (1999), 147–178.
- [17] K. McGerty, *cells in quantum affine sl_n* , Inter. Math. Research Letters **24** (2003), 1341–1361.
- [18] K. McGerty, *Generalized q -Schur algebras and quantum Frobenius*, Adv. **214** (2007), 116–131.
- [19] H. Nakajima, *Affine cellularity of quantum affine algebras*, Arxiv: 14061298.
- [20] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebras*, Duke Math. J. **100**(1999), 267–297.
- [21] D. Yang, *On the affine Schur algebra of type A* , Comm. Algebra, **37**(2009), 1389–1419.
- [22] D. Yang, *On the affine Schur algebra of type A II*, Algebra Rep Theory, **12**(2009), 63–75.
- [23] G. Yang, *Affine cellularity of $S_\Delta(2, 2)$* , arxiv: 1402.2715.
- [24] G. Yang, *Affine cellular algebras and Morita equivalences*, Arch. Math., **102**(2014), 319–327.

MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA.

E-mail address: bmdeng@math.tsinghua.edu.cn

SCHOOL OF SCIENCE, SHANDONG UNIVERSITY OF TECHNOLOGY, ZIBO 255049, CHINA

E-mail address: yanggy@mail.bnu.edu.cn